Maths for Computing
Assignment 5 Solutions

1. *(5 marks)* Prove that if P and Q are longest paths (of the same length) in a connected graph, then P and Q have at least one vertex in common. Give a detailed proof.
Solution: Let $P = \langle v_0, v_1, v_2, \ldots, v_n \rangle$ and $Q = \langle u_0, u_1, u_2, \ldots u_n \rangle$ be any two longest paths of length n. For the sake of contradiction, suppose P and Q do not have a common vertex. Further, assume that n is even (proof for odd n is similar).

Since the graph is connected, some vertex on P must have a path to some vertex on Q. Let R be a path from v_i on P to u_j on Q such that none of the non-ending vertices of R are on P or Q. Path P can be split into two paths, say S and T, from v_0 to v_i and v_i to v_n, respectively. We can say that one out of S and T will be of length at least $n/2$. Otherwise, the sum of the lengths of S and T, i.e., length of P, will be less than n, which is not possible. Similarly, Q can be split into two paths, say L and M, from u_0 to u_j and u_j to u_n, respectively. And one out of L and M must be of length at least $n/2$.

Suppose S and L are of length at least $n/2$. Then we can concatenate S, R, and $reverse(L)$ to create a path of length more than n, which is not possible as longest paths are of length n in G. Hence, a contradiction. Similarly, for other possibilities where S and M, T and L, or T and M are the paths of length at least $n/2$, we can create a path of length more than n.

2. *(5 marks)* Prove that if G is a disconnected graph, then $\overline{G}$ is connected.
Solution: Let $G_1, G_2, \ldots, G_k$ be the connected components of G. Now, in $\overline{G}$, consider any two vertices x and y. We show below that there will be a path between them.
Case 1: If x and y are in different components in G, then there cannot be an edge between them in G, and thus there will be an edge between them in $\overline{G}$. Hence, $\langle x, y \rangle$ will be a path between x and y.
Case 2: Suppose x and y are in the same component in G, say G_i. Since G is disconnected, there must be more than one component in G. Therefore, there will be some other component G_j with a vertex w in it such that there are no edges from x to w and y to w in G. Hence, in $\overline{G}$, there will be edges from x to w and y to w, creating a path from x to y.

3. *(5 marks)* Let M be a maximal matching and M' be any matching in a graph G. Prove that $|M| \geq |M'|/2$.

Solution: For the sake of contradiction, suppose $|M| < |M'|/2$.

Let $|M'| = k$ and $\{v_i, u_i\} \in M'$, for every $i \in [k]$. We claim that for some $j \in [k]$, both v_j and u_j are uncovered by M, and hence M is not a maximal matching as we can add $\{v_j, u_j\}$ to it. If the claim is not true, then for every $i \in [k]$, either v_i or u_i are covered by M. But this implies that M is covering at least k vertices, which is not possible as $|M| < |M'|/2 = k/2$ and a matching of size $< k/2$ can cover less than $2 * k/2 = k$ vertices.

4. *(5 marks)* Prove that a tree always has more leaves than vertices of degree three.
Solution: We will prove it using induction on the number of vertices.

Basis Step: For the tree of one vertex, the statement is trivially true.
Inductive Step: Let T be a tree of $k+1$ vertices. Let u be one of the leaves of T and v be its only neighbour. From inductive hypothesis, in $T-u$, the number of leaves, say j, is more than the number of vertices with degree three, say k, i.e., $j > k$.

We now argue that for T as well the statement is true by putting back u in $T - v$ to get T. Note here that T has only one more vertex, u, from $T - u$, and among the rest of the common vertices, degree of only v differs in T and $T - u$.
We divide the rest of the proof based on the degree of v in $T - u$.

Case 1: When degree of v is 0*:*
T in this case will have two more leaves, u and v, than $T - u$. Also the number of vertices with degree 3 will not change from $T - u$ to T. Therefore, the number of leaves in T is $j + 2$ and the number of vertices with degree 3 in T is k. Clearly, $j > k \implies j + 2 > k$.

Case 2: When degree of v is 1:
T in this case will have the same number of leaves as $T - u$ as v will not be a leave in T but u will be. Again the number of vertices with degree 3 will not change from $T - u$ to T. Therefore, the number of leaves in T is j and the number of vertices with degree 3 in T is k. We know that $j > k$.

Case 3: When degree of v is 2.
T in this case will have one more leave, u, than $T - u$. The number of vertices with degree 3 will increase by 1 from $T - u$ to T as v's degree will change from 2 to 3. Therefore, the number of leaves in T is $j + 1$ and the number of vertices with degree 3 in T is $k + 1$. Clearly, $j > k \implies j + 1 > k + 1$.

Case 3: When degree of v is 3.
T in this case will have one more leave, u, than $T - u$. The number of vertices with degree 3 will decrease by 1 from $T - u$ to T as v's degree will change from 3 to 4. Therefore, the number of leaves in T is $j + 1$ and the number of vertices with degree 3 in T is $k - 1$. Clearly, $j > k \implies j + 1 > k - 1$.

Case 3: When degree of v is ≥ 4.
T in this case will have one more leave, u, than $T - u$. The number of vertices with degree 3 will not change from $T - u$ to T. Therefore, the number of leaves in T is $j + 1$ and the number of vertices with degree 3 in T is k. Clearly, $j > k \implies j + 1 > k$.

5. *(5 marks)* Prove that Petersen graph does not contain two perfect matchings M and M' such that $M \cap M' = \emptyset$. You can use the results proved in class or tutorials without proving them again. *(Hint: The length of the smallest cycle in Petersen graph is* 5.*)*
Solution: Let M and M' be two disjoint matchings of the Petersen graph. Remember we proved in the tutorial that the graph made from the original vertices of the graph and edges of $M \oplus M'$ has either isolated vertices or even length cycles as its components. $M \oplus M'$ has 10 edges. We will now show that the graph of original vertices and edges $M \oplus M'$ cannot contain a cycle of length 2, 4, 6, 8, 10. Thus, it is not possible to have two disjoint matchings in Petersen graph.

Cycle of length 2 is not possible because parallel edges are not allowed in the definition of graphs. Cycle of length 4 is not possible as Petersen graph does not contain a cycle of length less than 5. *(You can use the hint without proving it, although proof is easy.)*

Cycle of length 6 is possible, but the other 4 edges have to form a cycle which is not possible. Similarly, cycle of length 8 is not possible as other two edges cannot form a cycle of length 2.

We now have to show that there is no cycle of length 10, i.e., there is no hamiltonian cycle in Petersen graph.

Suppose there is a hamiltonian cycle, say $C = \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1 \rangle$. Now Petersen graph contains 15 edges. Out of 15 edges, 10 edges are used in the hamiltonian cycle. We will prove now that we can not put the remaining 5 edges in the cycle without creating a cycle of length less than 5.

Let's name the remaining 5 edges as $e_1, e_2, e_3, e_4,$ and e_5. Suppose every edge of the remaining edges is connecting the opposite vertices, say e_1 connects v_1 and v_6, e_2 connects v_2 and v_7, e_3 connects v_3 and v_8, e_4 connects v_4 and v_9, and e_5 connects v_5 and v_{10}. In such a case, we can easily spot many 4 length cycles, such as $\langle v_1, v_6, v_5, v_{10}, v_1 \rangle$. Hence not all edges can connect opposite vertices of C.

Also, if some e_i connects two vertices who are at distance 3 or less in C, then that will also create a cycle of length less than 4.

Hence, there must be an edge e_k that connects two vertices, say v_i and v_j, that are at distance 4 in C. But now we cannot add an edge to the vertex opposite to v_j (or v_i), say v_k, without creating a cycle of length 4 or less. But there should be an edge apart from the two edges of hamiltonian cycle on v_k as degree of every vertex of Petersen graph is 3.

6. *(5 marks)* Petersen graph is non-planar. Prove it using Kuratowski's Theorem.
Solution: We can prove Petersen is non-planar by showing that it contains a subdivision of $K_{3,3}$. The below graph is clearly a subgraph of Petersen graph. It is also a subdivision of $K_{3,3}$. Red and green are the original vertices of $K_{3,3}$ such that there is an edge between every pair of red and green vertices. Black vertices are introduced by subdividing 4 out of 9 edges of the graph.